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# SOME FORMULATIONS OF BOUNDARY-VALUE PROBLEMS OF L-PLASTICITY

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1. The construction of mathematical models of a deformable medium is usually reduced to describing the relations between the stress and strain (velocity) tensors. Such an approach is based on two hypotheses: 1) The medium is assumed continuous; 2) in constructing the model any infinitesimal volume of the medium is allotted the properties of a macrospecimen if the latter is deformed under certain boundary conditions allowing a homogeneous stress distribution. Thus, if the specimen is deformed elastically, then it is assumed that each volume element is also deformed elastically. This assumption permits the description of the elasticity to reduce to the description of the elastic behavior of the volume element. By analogy, the legitimacy of such a transfer is also assumed in an investigation of the plastic behavior. Hence, as in the theory of elasticity, the problem of constructing a plastic model reduces to describing the plastic behavior of a volume element of a continuous medium.

However, a class of materials can be mentioned for which the hypothesis of identity between the properties of the specimen and its volume element is not satisfied even approximately. Indeed, let a certain specimen disclose plastic properties under definite loads. A situation is possible when the specimen is divided up into discrete slip surfaces on separate parts (blocks) under the loads mentioned. In this case the plastic properties of the specimen are entirely related to not only the inelastic strains of the blocks but also to their relative slips. If the blocks are deformed elastically, then the plastic properties of the specimen depend only on their relative slips.

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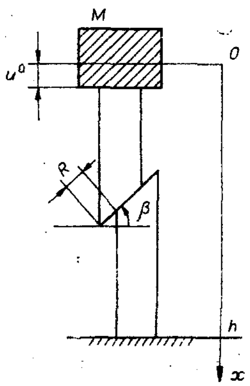


Fig. 1

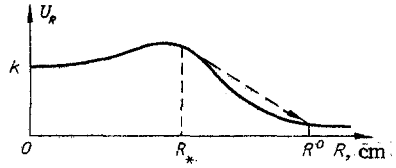


Fig. 2

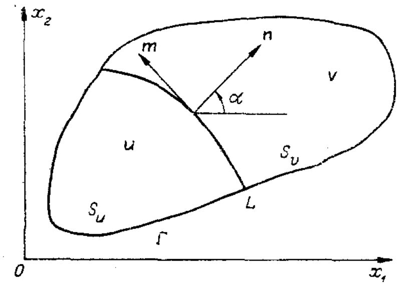


Fig. 3

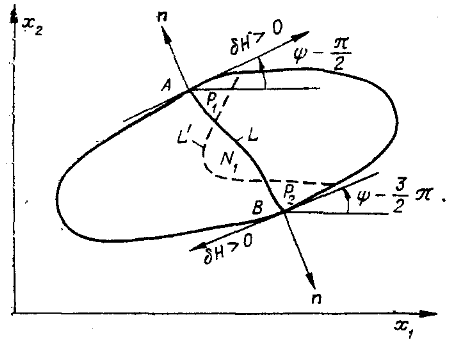


Fig. 4

Therefore, three stages of material deformation can be extracted in the general case: 1) elastic strain; 2) plastic strain; 3) strain under shear localization conditions. We designate material strain in the third stage as the L-plastic state.

Naturally, some of the strain stages can be missing for definite materials. Moreover, the realization of any stage depends on the specific loading conditions.

Methods for the theoretical analysis of the elastic and plastic strains are based on the two above-mentioned hypotheses and are close in the sense that the model construction is reduced in both cases to a description of the stress and strain (velocity) relations characterizing the state of the volume element. Passage to the volume element is impossible for the L-plastic stage of the strain and, hence, new problems arise in the description of L-plasticity: formulation of the criterion for the appearance of slip surfaces, determination of the relationship between the slip with a dimensionality of length, along the slip surface and the appropriate stresses; the description of the material properties on which the spacing between the slip surfaces can depend, etc. Experiments with different soils, friable bodies, mountain rocks, metals, et al. show that in many cases these materials behave as L-plastics; i.e., under sufficient stress the materials satisfy the imposed boundary conditions not only by the appearance of plastic domains, but also by slipping over separate surfaces [1-6]. The transfer to the L-plastic stage can be due to different causes: the losses of stability of the strain process without localization [6] (particularly under loading conditions allowing homogeneous strain [4]), initial inhomogeneity of the material (there is attenuation over isolated surfaces), etc.

The localization property played a fundamental part in the first stages of the development of soil mechanics. In particular, the limit condition of dry friction was introduced by Coulomb for localization surfaces, and was meaningful only on these surfaces. Then Cutter introduced the hypothesis about satisfying the Coulomb hypothesis at each point of the deformable mass. This hypothesis gave a thrust to new formulations and solutions of the problems in soil mechanics. At present, two directions are being developed in soil mechanics [7]. The results obtained within the framework of the first direction are considered approximate, engineering results, while the results obtained within the framework of the Cutter hypothesis are exact. However, it should be noted that accuracy in this case is kept in mind not in the sense of a more adequate description of the actual processes but in the sense of methods of solving the formulated mathematical problem. The approximate nature of the results of the first direction is not related to its principles but to the fact that within the framework of this direction material deformation outside the localization surfaces is not usually considered and the surfaces themselves are assumed known in advance with a certain arbitrariness. At the same time, within the framework of models based on the Cutter hypothesis, the mathematical problem is solved ordinarily without the introduction of additional hypotheses and, hence, the methods of solution and the results are considered exact. If the meaning of degree of adequacy in the description of the actual processes were to be ascribed to the accuracy concept, then in some cases the "approximate" solutions of the first direction can turn out to be much more accurate than the "exact" solutions of the second direction. Moreover, within the framework of the first direction, the part of the additional hypotheses can be reduced, and then the "accuracy" of the results of both directions will be comparable even in the sense of the rigor of the mathematical methods used.

The main reason whereby the opposition of these directions is not correct in the general case is related to the fact that different stages of material deformation are described within their frameworks: The Coulomb direction refers to the L-plastic stage of deformation and is substantially a set of engineering methods to solve L-plasticity problems, while the Cutter direction refers to the second plastic stage of deformation.

The facts of localization are not expressed as clearly in metals as in soils and, hence, the direction analogous to the Coulomb direction has practically not been developed in plasticity theory. The mathematical theory of plasticity from the time of its origin was developed along the second path, on the basis of the hypothesis of complying with some plasticity condition at every point of the deformable medium.

Meanwhile sufficiently many experimental facts exist on localization of shear strains in metals. These data show that the L-plastic stage of deformation plays an essential part even for metals, and the direction analogous to the Coulomb direction in soil mechanics should be developed in plasticity theory also.

The deformation stage in shear localization conditions is closely related to rupture. Rupture can provisionally be considered the last, fourth stage of material deformation. Rupture is understood to be the process of separating material into parts, which substantially influences the functional features of the material (the construction) because of the disappearance of interaction over the appropriate surfaces. The L-plastic stage denotes localization of the deformation under conditions when sufficiently strong interaction forces exist on the localization surfaces and the material (structure) as a whole does not alter its functional criteria. For instance, under definite conditions compression of specimens results in the appearance of domains of restricted deformation in the compressing slabs (compression nuclei, domains of direct influence of the slab), which are separated from domains abutting on the specimen free surface (domains of free surface influence) by certain localization surfaces [2] (the L-plastic stage of deformation). At this deformation stage, the specimen continues to exert resistance to compression and will outwardly behave exactly as under plastic compression. A further L-plastic deformation of the specimen results in the formation of new surfaces (vertical because of the disjoining effect of the domains abutting on the slabs, or oblique, sometimes coinciding with the localization surfaces) over which there is already no interaction between the material particles; the specimen alters its functional criteria. This is the rupture stage.

Therefore, there exists a broad class of problems in which the L-plastic stage of deformation plays a substantial part and should be taken into account in theoretical computations; L-plastic formulations are needed in problems of uniaxial and multiaxial compression, the imprint of a press, the insertion of a wedge, the dumping of welds and support walls in problems of the motion of materials in converging channels [6], the deformation of mountain rock around drifts, problems of pressure treatment of metals and powders, etc.

It follows from the definition that the L-plasticity models abut on the class of rupture models in which the interaction between crack edges is taken into account [8-11].

Let us examine certain formulations of the boundary value problems of L-plasticity in the case of plane strain or the plane stress state. The solution of the boundary value problems can be reduced to seeking discontinuous functions which bring an extremum to definite functionals which depend on both the behavior of the function in the smoothness domains and on the magnitudes of their discontinuities.

2. Such an approach can be illustrated by the following example. Let an elastic specimen under strain plane conditions be compressed by a heavy body M (Fig. 1). The height of the specimen is h, the width $l$, there is not displacement at the point $x=h$, but it equal $u^0$ at the point $x=0$. Let us assume that under certain boundary conditions a slip line with the slope $(\pi/2-\beta)$ to the specimen axis appears in the specimen. Let R denote the magnitude of the slip along a line. Then the total displacement of the point $x=0$ is comprised of two parts: the displacement $u_e$ because of elastic deformations of the specimen, and the displacement $R \sin \beta$ because of the slip; i.e.,

$$u^0 = u_e + R \sin \beta. \tag{2.1}$$

Let us assume the energy dissipation U per unit length of the slip line depends only on the magnitude of the slip R. Under active loading, the dissipation U(R) can provisionally be considered the "potential" energy of the body stored in the line of discontinuity if thermal effects are neglected. Then the total "potential" energy of the system specimen—heavy body has the form

$$W = mg\,[h - (u_e + R\sin\beta)] + \frac{W(R)}{\cos\beta} + \frac{1}{2}\frac{l}{h}\frac{E}{1-\nu^2}(u_e)^2, \tag{2.2}$$

where mg is the body weight; and E; $\nu$ are elastic constants. The energy W is a function of two variables: $u_e$ and R. Let us assume that only those of the all possible values of $u_e$, R are real which bring the energy W to a minimum. We shall later impose condition (2.1) on the variables $u_e$ and R, where the quantity $u^0$ (the loading parameter) is given. The requirement of stationarity of (2.2) results in a final equation in R,

$$U_R(R) = -\frac{\lambda}{h}R + \frac{\lambda}{h\sin\beta}u^0, \tag{2.3}$$

where $\lambda = \frac{E}{1-\nu^2}\sin^2\beta\cos\beta > 0$. The expression for the second variation shows that the equilibrium of (2.3) will be stable for $U_{RR} > -\lambda/h$ and unstable or indifferent in the remaining cases. The derivative $U_R$ has the meaning of a tangential stress on the slip line (Fig. 2). The condition relating the tangential stress to the slip with a length dimensionality was introduced in [9] and for a generalization of continual models [12, 13] in [14]. It is convenient to investigate the solution of (2.3) graphically on the $(U_R, R)$ plane. Let the loading parameter increase monotonically from zero. Then for $0 \le u^0 < u_1$ there are no shears along the slip line and the whole specimen is deformed entirely elastically. Under further loading $u_1 \le u^0 < u_2$ the slip line starts to function and the slips increase monotonically from 0 to $R_*$ (see Fig. 2). The specimen deformation is stable at this stage. Later, two substantially different strain modes are possible. If the specimen length is less than some critical ($h < -\lambda/U_{RR}$) value for given elastic properties of the specimen and the stress-slip diagram, then the strain on the descending branch will be stable; i.e., the slip R will increase monotonically for $u \ge u_2$ where a small increase in the slip R will correspond to a small increase in the loading parameter. For a sufficiently long specimen (or a sufficiently steep descending branch, if the specimen length and its elastic characteristics are fixed), the stability of the strain process is spoiled and the magnitude of the slip changes by a jump from $R_*$ to some value $R^0$ (see Fig. 2). Situations are also possible when the deformation is stable on some sections of the descending branch and unstable on others. All the parameters $u_1$, $u_2$, $R_*$, $R^0$ can be determined from (2.3).

Let us examine the mechanical meaning of instability. Two processes occur simultaneously on the descending branch: energy dissipation on the slip line and diminution of the potential elastic energy of the material off the line. If the energy liberated is less than is dissipated, then the strain process is stable and can be continued only under the supply of energy from outside. If the energy liberated is greater than is dissipated, then the strain is unstable and the slips increase because of the internal resources of the material. Part of the energy hence goes over into kinetic energy. Analogous effects hold in the continual model also [14]. In principle, the realization of such an experiment with controllable displacements on the boundary will allow the determination of the diagram U(R).

Let us note that all the deductions considered follow from the variational principle taken and could be obtained by direct methods for the minimization of the energy W. Therefore, the application of a variational principle permits determination of the instant at which shears start along the slip line, solution of the problem of stable development of the slip line, determination of the loading parameter for which unstable development starts, calculation of the part of the elastic energy which goes over into kinetic energy during the unstable process, and consideration of the further strain when the slip again goes over into the stable mode. The possibilities mentioned are conserved even in the general situation of plane and three-dimensional strain.

3. Let us consider the plane strain of an L-plastic material. Let L be the line of a possible discontinuity, **n**, **m** the normal and tangent vectors to the line, and $\alpha$ the angle between the vector **n** and the axis $Ox_1$. Let $u_1$, $u_2$, $v_1$, $v_2$ denote components of the displacement vector in the domains $S_u$, $S_v$ (Fig. 3). The normal component of the displacement is continuous on the line L,

$$(v_1 - u_1)\cos\alpha + (v_2 - u_2)\sin\alpha = 0, \tag{3.1}$$

but the tangential component can experience a discontinuity R:

$$R = -(v_1 - u_1)\sin\alpha + (v_2 - u_2)\cos\alpha. \tag{3.2}$$

We shall designate the magnitude of the discontinuity the slip. Let us assume that the energy dissipation U per unit length of the slip line only depends on the magnitude of the slip (3.2) and the coordinates: U = U(R,

$x_1, x_2$). Let $F(u_i, p_{ij}, x_i)$ and $F(v_i, q_{ij}, x_i)$ denote the potential energy of unit "volume" of material outside the slip line. Here $i, j = 1, 2$; $p_{ij} = \partial u_i/\partial x_j$; $q_{ij} = \partial v_i/\partial x_j$, $F_{p_{12}} \equiv F_{p_{21}}$, $F_{q_{12}} \equiv F_{q_{12}}$. On certain sections $\Gamma_u$, $\Gamma_v$ of the outer boundary of the domains $S_u$, $S_v$ let the stress vector $\{X_i\}$ be given as a function of the boundary displacements, where a potential $\Phi(u_1, u_2)$: $d\Phi = X_1(u_1, u_2)du_1 + X_2(u_1, u_2)du_2$ exists. An analogous condition is also satisfied on $\Gamma_v$. The case when the functions $X_i(u_j)$ reduce to a constant corresponds to given boundary stresses. Then the total "potential" energy of the system has the form

$$W = \int_{S_u} F(u_i, p_{ij}, x_i)\,ds + \int_{S_v} F(v_i, q_{ij}, x_i)\,ds + \int_L U(R, x_i)\,dl - \int_{\Gamma_u} \Phi(u_i)\,dl - \int_{\Gamma_v} \Phi(v_i)\,dl. \tag{3.3}$$

If there are several lines of possible slip, then the appropriate summations should be in (3.3). To cut down the writing, let us limit ourselves to the case of one line. The results are analogous in the general case. The expression (3.3) is a functional in the line of possible discontiniuty L and the displacement field. Let us first assume that the line L is known from additional considerations and is not varied. Let us introduce the following variational principle: Out of all the kinematically possible displacement fields satisfying the continuity condition (3.1) and the boundary conditions in the displacements, that which makes the functional W a minimum will be real. The necessary condition for minimality results in the following equations in the domains $S_u$, $S_v$:

$$F_{u_i} - \frac{\partial}{\partial x_j}\{F_{p_{ij}}\} = 0_i, \quad F_{v_i} - \frac{\partial}{\partial x_j}\{F_{q_{ij}}\} = 0_i, \tag{3.4}$$

the boundary conditions on $\Gamma_u$, $\Gamma_v$:

$$\begin{gathered} F_{p_{i1}}\cos\psi + F_{p_{i2}}\sin\psi = X_i(u_1, u_2), \\ F_{q_{i1}}\cos\psi + F_{q_{i2}}\sin\psi = X_i(v_1, v_2), \end{gathered} \tag{3.5}$$

and the conditions on the line of possible discontinuity:

$$F_{p_{i1}}\cos\alpha + F_{p_{i2}}\sin\alpha = F_{q_{i1}}\cos\alpha + F_{q_{i2}}\sin\alpha = \Sigma_i; \tag{3.6}$$

$$-\sin\alpha\Sigma_1 + \cos\alpha\Sigma_2 = U_R, \tag{3.7}$$

where the braces denote the total derivative; the subscript in the right side of (3.4) shows that the summation is carried out only over the subscript j; $\psi$ is the angle between the normal to the outer contour and the axis $Ox_1$. Equations (3.4) are the equilibrium equations in displacements; (3.5) are the boundary conditions for the stresses, and (3.6) is the continuity condition for the tangential and normal stress tensor components to the line L. The limit of the tangential stress in the domains $S_u$, $S_v$ upon approaching the line L is on the left in condition (3.7), while the tangential stress $U_R = T$, which is generated on the line because of the slip R, is on the right.

Therefore, minimization of the functional W under the condition of continuity of the normal displacement results in a closed system of equations, natural boundary conditions, and natural conditions for continuity on the line of possible slip. The corollaries obtained verify the applicability of the variational principle introduced.

The formulation considered above is substantially semiinverse since the lines of possible slip are assumed known either from experimental data or from symmetry conditions or from additional considerations. In an exact formulation the lines should be determined during solution of the problem. If the line is not fixed in advance, then the total "potential" energy W is a functional of the displacement field and the line of possible discontinuity. The minimum of the functional which is reached in the displacement field (3.4) is a functional in the line of possible slip. It is natural to consider such lines that reduce the total "potential" energy to the deepest minimum. In this case the problem reduces to seeking the minimum of the functional W under conditions when both the displacement field and the position of the slip line admit variation.

Let $x_i = x_i(t)$, $t \in [t_1, t_2]$ be the parameteric equations of the curve $L$, $a = \sqrt{(x_1')^2 + (x_2')^2}$. The symbol $\delta$ denotes the variation of the functions for fixed arguments and the symbol $\delta$ the variation under the condition that the arguments are also varied. We denote the variations of the arguments themselves by $\bar{\delta}x_i$. We consider the varied line L' close to L in the sense of a first-order nearness ($\bar{\delta}x_i \ll 1$, $\bar{\delta}x_i' \ll 1$). We denote the varied domains $S_u$, Sv by $S_u'$, $S_v'$. Each of the intersections $(S_u' \cap S_v)$ $(S_v' \cap S_u)$ decomposes into a number of simply connected domains $P_k$, $N_k$ (Fig. 4). We extract the sums of double integrals over the domains $P_k$, $N_k$ from the variation $\delta W$ and convert them to the form

$$\sum_k \int_{P_k} [F(u_i \ldots) - F(v_i \ldots)]\,ds + \sum_k \int_{N_k} [F(v_i \ldots) - F(u_i \ldots)]\,ds = \int_{t_1}^{t_2} Q(\cos\alpha\bar{\delta}x_1 + \sin\alpha\bar{\delta}x_2)\,a(t)\,dt,$$

where $Q = F(u_i \ldots) - F(v_i \ldots)$. Let us examine the constraints on the line of possible slip. The continuity condition (3.1) results in a relation on the variation

$$(\bar{\delta} v_1 - \bar{\delta} u_1)\cos\alpha + (\bar{\delta} v_2 - \bar{\delta} u_2)\sin\alpha + R\,\bar{\delta}\alpha = 0, \tag{3.8}$$

where $\bar{\delta}\alpha = \dfrac{x_1'\bar{\delta}x_2' - x_2'\bar{\delta}x_1'}{(x_1')^2 + (x_2')^2}$, and

$$\bar{\delta} u_i = \delta u_i + p_{ir}\bar{\delta} x_r, \quad \bar{\delta} v_i = \delta v_i + q_{ir}\,\bar{\delta} x_r, \tag{3.9}$$

and the values of all the functions can be taken on L. Equations (3.8) and (3.9) in combination with the equation $-(\bar{\delta} v_1 - \bar{\delta} u_1)\sin\alpha + (\bar{\delta} v_2 - \bar{\delta} u_2)\cos\alpha = \bar{\delta} R$ form a closed system in the difference $(\delta v_i - \delta u_i)$. Let us convert the curvilinear integral over L into an ordinary definite integral. Then W can be considered as a functional in $v_i(x_j)$, $u_i(x_j)$ and $x_i(t)$, where outside the slip lines all the variations $\delta u_i$, $\delta v_i$ are independent, while related to one condition [(3.8)] on the line. Therefore, only five of the six variations $\delta u_i$, $\delta v_i$, $\bar{\delta} x_i$ will be independent on the line. It is convenient to select the variations $(\delta u_i + \delta v_i)$, $\bar{\delta} x_i$, and $\delta R$ as independent. Omitting further calculations, we present the final results. The extremum of the total "potential" energy is reached if the discontinuous displacement field satisfies the relationships (3.4)-(3.7), and the line of discontinuity satisfies the equations

$$\begin{aligned} &\frac{d}{adt}\left(\textstyle\sum_n R\cos\alpha + U\sin\alpha\right) + U_{x_1} + Q\cos\alpha + \Lambda_1 = 0, \\ &\frac{d}{adt}\left(\textstyle\sum_n R\sin\alpha - U\cos\alpha\right) + U_{x_2} + Q\sin\alpha + \Lambda_2 = 0, \end{aligned} \tag{3.10}$$

where

$$\Sigma_n = \Sigma_1\cos\alpha + \Sigma_2\sin\alpha;\ \Lambda_1 = (q_{11} - p_{11})\Sigma_1 + (q_{21} - p_{21})\Sigma_2;$$
$$\Lambda_2 = (q_{12} - p_{12})\Sigma_1 + (q_{22} - p_{22})\Sigma_2.$$

Equations (3.10) have been obtained as a result of equating the coefficients of the variations $\bar{\delta} x_1$, $\bar{\delta} x_2$ to zero. Since both variations are considered independent for $t_1 < t < t_2$, then information about the variation of the line of discontinuity along it should be contained in (3.10) also. It is natural that no conditions should be obtained on L in such a variation. Indeed, the system (3.10) can be converted to the form

$$\begin{aligned} &\frac{d}{adt}(R\Sigma_n) + U\frac{d\alpha}{adt} + U_{x_1}\cos\alpha + U_{x_2}\sin\alpha + Q + \Lambda_1\cos\alpha + \Lambda_2\sin\alpha = 0, \\ &-\frac{dU}{adt} + R\Sigma_n\frac{d\alpha}{adt} - U_{x_1}\sin\alpha + U_{x_2}\cos\alpha - \Lambda_1\sin\alpha + \Lambda_2\cos\alpha \equiv 0, \end{aligned} \tag{3.11}$$

where it can be shown by direct confirmation that the last equality is satisfied identically.

Let us examine the natural boundary conditions for the line (3.11). Let us first note that the ends of the line should lie on the outer contour, i.e., at the points $t = t_1, t_2$,

$$\bar{\delta} x_1 = \bar{\delta} H\sin\psi, \quad \bar{\delta} x_2 = -\delta H\cos\psi, \tag{3.12}$$

where $\delta H > 0$ corresponds to shifting the point $t_1$ towards the domain $S_u$ and the point $t_2$ towards the domain $S_v$ (see Fig. 4). Members representing the value of certain functions at the points $t_1$, $t_2$ can be extracted from the expression for the variation of the total "potential" energy W. The requirement of minimality of the functional W under the constraints (3.12) results in definite boundary conditions at the points $t_1$, $t_2$. Let us consider the condition at the point $t_2$. The results are analogous for the point $t_1$. It follows from the condition $\delta W = 0$ that for $t = t_2$, $\delta H \neq 0$,

$$-[U(R, x_1, x_2)\cos(\psi - \alpha) + \Sigma_n R\sin(\psi - \alpha)] + c = 0, \tag{3.13}$$

where c is a component referred to $\delta H$ which can appear in the calculation of the variation

$$\delta I_0 = -\int_{\Gamma_u'} \Phi(u_i + \delta u_i)\,dl + \int_{\Gamma_u} \Phi(u_i)\,dl - \int_{\Gamma_v'} \Phi(v_i + \delta v_i)\,dl + \int_{\Gamma_v} \Phi(v_i)\,dl,$$

where $\Gamma_u'$, $\Gamma_v'$ are contours of $\Gamma_u$, $\Gamma_v$ after the variation of the line L. Let continuous displacements be given in a two-sided neighborhood of the point $t_2$. Then $\Gamma_u' = \Gamma_u$, $\Gamma_v' = \Gamma_v$, and $c = 0$. If continuous stresses or the potential $\Phi$ are given in the two-sided neighborhood of the point $t_2$, then $c = -\Phi(u_i) + \Phi(v_i)$. Let us now assume that the position of the end of the slip line L is known (e.g., $t_2$ coincides either with a point of discontinuity of the boundary displacements or stresses, or with an interchange point of the type of the boundary conditions, etc.). In this case $\delta H \equiv 0$ and condition (3.13) is replaced by the following: $x_i(t_2) = x_i^0$, where the values of $x_i^0$ are given.

Therefore, the strengthened variational principle permits determination of both the discontinuous displacement field and the position of the line of discontinuity. However, in the general case the lines obtained directly from the variational principle cannot be considered real since the loading history of the material is not taken into account in such an approach. Thus, in the case of uniaxial compression, the slope of the slip line equals $\pi/4$ only for $mg = 2kl$. If $\beta$ is evaluated directly for $mg > 2kl$ without taking account of the gradual increase in mg, then it turns out that the angle $\beta \neq \pi/4$ and depends on mg. Nevertheless, the lines (3.11) can be used to estimate the actual process of L-plastic deformation.

The total "potential" energy was used above as the functional. Analogous formulations for other functionals as well (of the type of extra work, etc.) can also be considered. All the results are easily extended to the case when in addition to the tangential, a discontinuity in the displacement component normal to the line is also allowed (the problem of the development of cracks of a normal discontinuity, etc.) [15]. Moreover, analogous results hold even in the case of three-dimensional strain (the displacement discontinuities are allowed on isolated surfaces).

4. In conclusion, let us examine the case when the L-plasticity problem can be reduced to a boundary value problem for analytic functions. Let the domains outside the slip line be deformed linearly–elastically, let $\varphi$, $\psi$ be complex potentials in the domain $S_u$, and $\xi$, $\eta$ potentials in the domain $S_v$. Let the primes denote the displacement and stress components in local coordinates with directions of the axes along the vectors **n**, **m** (see Fig. 3). Then

$$2u_1' = (u_1 + iu_2)\, e^{-i\alpha} + (u_1 - iu_2)\, e^{i\alpha},$$
$$2\sigma_{11}' + 2i\sigma_{12}' = (\sigma_{11} + \sigma_{22}) + (\sigma_{11} - \sigma_{22} + 2i\sigma_{12})\, e^{-2i\alpha}. \tag{4.1}$$

Analogous formulas are true in the domain $S_v$ also. The expression for the jump in the displacement component tangent to the slip line $R = v_2' - u_2'$ with the continuity condition $u_1' = v_1'$ taken into account can be converted to the form

$$iR = (v_1 + iv_2)e^{-i\alpha} - (u_1 + iu_2)e^{-i\alpha}. \tag{4.2}$$

The inverse function can be constructed by means of the known function $U_R$ $(R, x_i)$ and transformed to the function $\Omega$:

$$iR = \Omega\left(2i\sigma_{12}', z\right), \qquad z = x_1 + ix_2. \tag{4.3}$$

Therefore, the continuity conditions in the left sides of (4.1) and the conjugate condition (4.3) should be satisfied on the line of possible slip. Let us express the displacement and stress components in the domains $S_u$, $S_v$ by the Kolosov–Muskhelishvili formulas [16], and let us use the relationships (4.1)-(4.3). Then all the conjugate conditions on the line L can be represented in the form

$$\begin{gathered}
(\varkappa\varphi - z\overline{\varphi'} - \overline{\psi})\, e^{-i\alpha} + (\varkappa\overline{\varphi} - \overline{z}\varphi' - \psi)\, e^{i\alpha} = \\
= (\varkappa\xi - z\overline{\xi'} - \overline{\eta})\, e^{-i\alpha} + (\varkappa\overline{\xi} - \overline{z}\xi' - \eta)\, e^{i\alpha}, \\
(\varphi' + \overline{\varphi'}) - (z\overline{\varphi''} + \overline{\psi'})\, e^{-2i\alpha} = \\
= (\xi' + \overline{\xi'}) - (z\overline{\xi''} + \overline{\eta'})\, e^{-2i\alpha}, \\
[\varkappa(\xi - \varphi) - z(\overline{\xi'} - \overline{\varphi'}) - (\overline{\eta} - \overline{\psi})]\, e^{-i\alpha} = \\
= 2\mu\Omega\,[-(z\overline{\varphi''} + \overline{\psi'})\, e^{-2i\alpha} + (\overline{z}\varphi'' + \psi')\, e^{2i\alpha},\ z],
\end{gathered} \tag{4.4}$$

where $\mu$ is the shear modulus, $\varkappa = (3-\nu)/(1+\nu)$ for the plane stress state and $\varkappa = 3 - 4\nu$ for plane strain, and $\alpha$ is a known function (semiinverse formulation, the lines of possible slip are known).

Therefore, the L-plasticity problem reduces to seeking the analytic functions $\varphi, \psi, \xi, \eta$, satisfying boundary conditions on the outer contour which are defined by the Kolosov–Muskhelishvili equations, and the conjugate conditions (4.4) on the line L.

The problem is also formulated analogously in the case of several lines of possible slip. An example of solving the boundary value problem is considered in [17].

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